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Regge Poles in πN Scattering and in $\pi+\pi \rightarrow N+\bar{N}$†

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We introduce the Regge-Froissart continuations of various partial-wave amplitudes for πN scattering into the complex J plane. The notion of J parity is clarified by considering parity nonconservation. The analyticity and symmetry properties of the Regge-Froissart continuation in the energy plane are also studied and the results of analysis applied to backward pion-proton scattering. A similar discussion is given of the $\pi+\pi \rightarrow N+\bar{N}$ channel and the forward elastic pion-proton scattering.

I. INTRODUCTION

THE importance of regarding the scattering amplitudes as a simultaneous analytic function of energy and angular momentum J was first pointed out by Regge for nonrelativistic potential scattering.[1] This notion has been extended to the relativistic S matrix and has already revolutionized present thinking in strong-interaction physics.[2] Here, we present a systematic discussion of the pion-nucleon problem from that point of view.

In Sec. II, we introduce the proper Regge-Froissart continuations of various partial-wave amplitudes into the complex J plane for πN scattering, where we assume parity nonconservation. This is done to elucidate the nature of the J parity and to bring out clearly that J parity has nothing to do with space parity. As a by-product of this discussion we clarify the concept of the range of exchange potential for the scattering of two unequal-mass particles—this is discussed in the Appendix. These J-plane continuations are studied in Sec. III as to their analytic behavior in the energy variable and new amplitudes free from kinematical singularities are introduced. Also, these amplitudes have important symmetry properties, which reflect in the expressions given in Sec. IV for the backward pion-proton scattering in the direct channel. The observed particle and resonance states in the πN channel are also discussed in Sec. III. The last two sections, V and VI, deal, respectively, with the J-plane analyticity of helicity amplitudes in the $\pi+\pi \rightarrow N+\bar{N}$ channel and with its implications for the forward elastic pion-proton scattering.

II. REGGE-FROISSART CONTINUATION OF PARTIAL-WAVE AMPLITUDES IN THE πN SCATTERING CHANNEL: J PARITY

We introduce the proper analytic continuations into the complex J plane of the various partial-wave amplitudes for πN scattering. Even though parity is conserved, the discussion is carried out for the general parity-nonconserving case because confusion has prevailed whether J parity and ordinary parity (i.e., space parity) are distinct quantum numbers for Regge trajectories. The J parity is the notion that only the alternate physical J values on the Regge trajectories give rise to physical bound states and resonances. This certainly is true for spin-zero–spin-zero particle scattering. Unfortunately, the separation of the amplitude into even and odd J-parity parts for this case coincides with the separation into even and odd space-parity parts. So one is likely to regard the J-parity notion as nothing distinct from the space-parity diagonalization, and this is the source of confusion. The only way to resolve this situation is to study a problem in which parity is not conserved and then see whether one still has the notion of J parity. As parity conservation is implied by angular-momentum conservation for scattering of two spin-zero particles, one has to study a problem with spin. In the following we study scattering of a spin-zero particle by a spin-one-half particle; i.e., we study πN scattering where we assume parity nonconservation.

There are now four independent invariant amplitudes, instead of the usual two amplitudes A and B. The T matrix can be expressed as

$$T=-A+i\gamma\cdot QB+i\gamma_5\gamma\cdot QC-\gamma_5 D, \qquad (2.1)$$

where

$$Q=\tfrac{1}{2}(K_1+K_2),$$

and K_1 and K_2 are the four-momenta of the initial and the final pion, respectively.

The differential cross section $d\sigma/d\Omega$ can be written as

$$\frac{d\sigma}{d\Omega}=\sum_{\text{spins}} |\langle\text{final}|f|\text{initial}\rangle|^2, \qquad (2.2)$$

where

$$f=f_1+\sigma\cdot\hat{k}_f\sigma\cdot\hat{k}_i f_2+\sigma\cdot\hat{k}_f f_3+\sigma\cdot\hat{k}_i f_4, \qquad (2.3)$$

and $\hat{k}_f$ and $\hat{k}_i$ are unit vectors in the direction of the final and the initial pion three-momentum, respectively.

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[1] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947 (1960); T. Regge, A. M. Longoni, and A. Bottino, *ibid.* **23**, 954 (1962).

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1961); G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, *ibid.* **126**, 2204 (1962); R. Blanckenbecler and M. L. Goldberger, *ibid.* **126**, 766 (1962); B. M. Udgaonkar, Phys. Rev. Letters **8**, 142 (1962).

The f_i's are given by

$$f_1=[(E+m)/8\pi W][A+(W-m)B],$$
$$f_2=[(E-m)/8\pi W][-A+(W+m)B],$$
$$f_3=-(k/8\pi W)[WC+D],$$

and

$$f_4=-(k/8\pi W)[WC-D], \tag{2.4}$$

with k, E, and W being the magnitude of the three-momentum of the pion, the energy of the nucleon, and the total energy, respectively, in the center-of-mass system: i.e.,

$$E=(W^2+m^2-1)/2W,$$

and

$$4k^2=W^2-(2m^2+2)+(m^2-1)^2/W^2.$$

We might note that time-reversal invariance implies that $D=0$; i.e., that $f_3=f_4$. However, we do not need to assume time-reversal invariance.

The partial-wave decompositions of the f_i's are given by

$$f_1=\sum a_{J-\frac{1}{2},J-\frac{1}{2}}{}^J P_{J+\frac{1}{2}}'-\sum a_{J+\frac{1}{2},J+\frac{1}{2}}{}^J P_{J-\frac{1}{2}}',$$
$$f_2=\sum a_{J+\frac{1}{2},J+\frac{1}{2}}{}^J P_{J+\frac{1}{2}}'-\sum a_{J-\frac{1}{2},J-\frac{1}{2}}{}^J P_{J-\frac{1}{2}}',$$
$$f_3=\sum a_{J+\frac{1}{2},J-\frac{1}{2}}{}^J P_{J+\frac{1}{2}}'-\sum a_{J-\frac{1}{2},J+\frac{1}{2}}{}^J P_{J-\frac{1}{2}}',$$

and

$$f_4=\sum a_{J-\frac{1}{2},J+\frac{1}{2}}{}^J P_{J+\frac{1}{2}}'-\sum a_{J+\frac{1}{2},J-\frac{1}{2}}{}^J P_{J-\frac{1}{2}}', \tag{2.5}$$

where $a_{L'',L'}{}^J$ is the partial-wave amplitude for transition between an initial and a final state, both with total angular momentum J, and having orbital angular momenta L' and L'', respectively. In the conventional notation $a_{J-\frac{1}{2},J-\frac{1}{2}}{}^J=f_{J-\frac{1}{2},+}$ and $a_{J+\frac{1}{2},J+\frac{1}{2}}{}^J=f_{J+\frac{1}{2},-}$. The summation over J runs over $J=1/2, 3/2, 5/2, \cdots$. The argument of the Legendre polynomials is

$$z=\cos\theta=1+(t/2k^2)=1+(2m^2+2-s-u)/(2k^2), \tag{2.6}$$

where s, u, and t are the usual invariant variables, which have the significance of becoming the total energy squared in the barycentric systems of the πN scattering channel, the crossed πN scattering channel, and the $\pi+\pi\rightarrow N+\bar{N}$ channel, respectively.

The projection formulas for the different partial-wave amplitudes can be worked out and are given by

$$a_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^J=\frac{1}{2}\int_{-1}^{+1}d(\cos\theta)[f_1P_{J\mp\frac{1}{2}}+f_2P_{J\pm\frac{1}{2}}],$$

and

$$a_{J\pm\frac{1}{2},J\mp\frac{1}{2}}{}^J=\frac{1}{2}\int_{-1}^{+1}d(\cos\theta)[f_3P_{J\mp\frac{1}{2}}+f_4P_{J\pm\frac{1}{2}}]. \tag{2.7}$$

Using these projection formulas (2.7), and the expressions (2.4) for f_i's in terms of the invariant amplitudes, we get, finally,

$$a_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^J=\frac{E+m}{16\pi W}[A_{J\mp\frac{1}{2}}+(W-m)B_{J\mp\frac{1}{2}}]$$
$$+\frac{E-m}{16\pi W}[-A_{J\pm\frac{1}{2}}+(W+m)B_{J\pm\frac{1}{2}}],$$

and

$$a_{J\pm\frac{1}{2},J\mp\frac{1}{2}}{}^J=-\frac{k}{16\pi W}[W(C_{J\mp\frac{1}{2}}+C_{J\pm\frac{1}{2}})$$
$$+(D_{J\mp\frac{1}{2}}-D_{J\pm\frac{1}{2}})], \tag{2.8}$$

where

$$A_J=\int_{-1}^{+1}d(\cos\theta)A(s,u,t)P_J(\cos\theta). \tag{2.9}$$

The B_J, C_J, and D_J are defined similarly.

Expressions (2.8) and (2.9) define the various partial-wave amplitudes for physical values of J. One has now to find an analytic continuation of these amplitudes into the complex J plane, from these physical J values, that is suitable for a Sommerfeld-Watson transform. As the only J dependence of partial-wave amplitudes is contained in A_J, B_J, C_J, and D_J, the problem reduces to finding a proper continuation of these quantities. To that purpose, we notice that invariant amplitudes satisfy fixed energy-dispersion relations of the type

$$A(s,u,t)=\frac{1}{\pi}\int_4^\infty\frac{A_t(s,t')dt'}{t'-t}+\frac{1}{\pi}\int_{(m+1)^2}^\infty\frac{A_u(s,u')du'}{u'-u} \tag{2.10}$$

$$=\frac{1}{\pi}\int_4^\infty\frac{A_t(s,x')dx'}{x'+2k^2(1-\cos\theta)}$$
$$+\frac{1}{\pi}\int_{(m+1)^2-(m^2-1)^2/s}^\infty\frac{A_u(s,x'+(m^2-1)^2/s)dx'}{x'+2k^2(1+\cos\theta)}. \tag{2.11}$$

By substituting expression (2.11) for A in (2.9), we get

$$A_{J-\frac{1}{2}}=\frac{1}{\pi k^2}\int dx'\,[A_t(s,x')$$
$$+(-)^{J-\frac{1}{2}}A_u(s,\,x'+(m^2-1)^2/s)]$$
$$\times Q_{J-\frac{1}{2}}\left(1+\frac{x'}{2k^2}\right). \tag{2.12}$$

We see that except for the $(-)^{J-\frac{1}{2}}$ factor, Eq. (2.12) provides an expression suitable for Sommerfeld-Watson transform. The canonical way to get rid of the $(-)^{J-\frac{1}{2}}$ factor is to define two analytic continuations of $A_{J-\frac{1}{2}}$, one away from even integral values of $J-\frac{1}{2}$, and another from odd integral values[3] of $J-\frac{1}{2}$; i.e.,

$$A_{J-\frac{1}{2}}{}^e=\frac{1}{\pi k^2}\int[A_t(s,x')+A_u(s,\,x'+(m^2-1)^2/s)]dx'$$
$$\times Q_{J-\frac{1}{2}}\left(1+\frac{x'}{2k^2}\right), \tag{2.13}$$

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[3] E. J. Squires, Nuovo Cimento **25**, 242 (1962). Also there is a good discussion of J parity or signature in the paper of S. C. Frautschi, M. Gell-Mann and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).

and

$$A_{J-\frac{1}{2}}{}^{\phi}=\frac{1}{\pi k^2}\int[A_t(s,x')-A_u(s,x'+(m^2-1)^2/s)]dx'\times Q_{J-\frac{1}{2}}\left(1+\frac{x'}{2k^2}\right).$$

If one uses $A_{J\mp\frac{1}{2}}{}^{e}$ for $A_{J\mp\frac{1}{2}}$ in Eq. (2.8), together with $B_{J\mp\frac{1}{2}}{}^{e}$ for $B_{J\mp\frac{1}{2}}$, $C_{J\mp\frac{1}{2}}{}^{e}$ for $C_{J\mp\frac{1}{2}}$, and $D_{J\mp\frac{1}{2}}{}^{e}$ for $D_{J\mp\frac{1}{2}}$, one obtains a continuation of the different partial-wave amplitudes that agrees with the amplitude for these physical values of $J=1/2, 5/2, 9/2, \cdots$ etc. Similarly, by using $A_{J\mp\frac{1}{2}}{}^{\phi}$ for $A_{J\mp\frac{1}{2}}$ in Eq. (2.8), and replacements $B_{J\mp\frac{1}{2}}{}^{\phi}$ for $B_{J\mp\frac{1}{2}}$, $C_{J\mp\frac{1}{2}}{}^{\phi}$ for $C_{J\mp\frac{1}{2}}$, and $D_{J\mp\frac{1}{2}}{}^{\phi}$ for $D_{J\mp\frac{1}{2}}$, one would obtain another continuation which agrees with the amplitude for these physical values of $J=3/2, 7/2, 11/2, \cdots$. This is precisely the notion of J parity, which here follows irrespective of any parity-conservation considerations, and only arising because of the simultaneous presence of the direct and exchange forces. It is a straightforward matter to express the Sommerfeld-Watson transform of the scattering amplitudes in terms of these even and odd J-parity continuations of the various partial-wave amplitudes. For example,

$$\begin{aligned}f_{1,2}=&\pm\frac{i}{4}\int_C\frac{dJ}{\cos\pi J}a_{J-\frac{1}{2},J-\frac{1}{2}}{}^{J,e}[P_{J\pm\frac{1}{2}}'(-z)\pm P_{J\pm\frac{1}{2}}'(z)]\\&\pm\frac{i}{4}\int_C\frac{dJ}{\cos\pi J}a_{J-\frac{1}{2},J-\frac{1}{2}}{}^{J,\phi}[P_{J\pm\frac{1}{2}}'(-z)\mp P_{J\pm\frac{1}{2}}'(z)]\\&\mp\frac{i}{4}\int_C\frac{dJ}{\cos\pi J}a_{J+\frac{1}{2},J+\frac{1}{2}}{}^{J,e}[P_{J\mp\frac{1}{2}}'(-z)\mp P_{J\mp\frac{1}{2}}'(z)]\\&\mp\frac{i}{4}\int_C\frac{dJ}{\cos\pi J}a_{J+\frac{1}{2},J+\frac{1}{2}}{}^{J,\phi}[P_{J\mp\frac{1}{2}}'(-z)\pm P_{J\mp\frac{1}{2}}'(z)],\end{aligned}\tag{2.14}$$

where on the right-hand side of Eq. (2.14) the upper signs refer to f_1, the lower signs to f_2, and C is the usual undistorted contour for the Sommerfeld-Watson transform. Of course, the contour C can be distorted in that entire region of the J plane over which the various continuations exist if the contribution of the enclosed singularities, in particular Regge poles, is included.

III. REGGE TRAJECTORIES IN THE πN SCATTERING CHANNEL

We now study the analyticity and symmetry properties in the energy variable of the continuation of the partial-wave amplitudes into the complex J plane, which we introduced in Sec. II. We also discuss the observed particle and resonance states in this channel in terms of the present analysis.

If we consider the different partial-wave amplitudes continued into the complex J plane as functions of the invariant variable $s=W^2$, we encounter, apart from the s-plane singularities of $A_{J\mp\frac{1}{2}}{}^{e,\phi}$, etc., kinematical singularities of the $\sqrt{s}$ type caused by the factors of $W-m$, $W+m$, and W, etc., which occur in the problem because of the spin. The existence of kinematical singularities in the s plane was already brought out for partial-wave amplitudes corresponding to the physical J values, by earlier authors.[4] Thus it is advantageous to work in the W-complex plane.

The functions $A_{J\mp\frac{1}{2}}{}^{e,\phi}$, etc., have additional branch points whose locations are given by $k^2=0$, when $J\mp\frac{1}{2}$ is not a positive integer, apart from the usual branch points, which are those of $A_{J\mp\frac{1}{2}}$ for physical values of J. However, the function $A_{J\mp\frac{1}{2}}{}^{e,\phi}/(2k^2)^{J\mp\frac{1}{2}}$ has precisely the same analytic structure as the function $A_{J\mp\frac{1}{2}}/(2k^2)^{J\mp\frac{1}{2}}$, for physical values of J.

We are thus led to consider the following quantities, if we wish to avoid any kinematical singularities:

$$h_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e,\phi)}(W)=\frac{16\pi W}{E\pm m}\frac{a_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e,\phi)}}{(2k^2)^{J-\frac{1}{2}}},\tag{3.1}$$

$$\begin{aligned}h_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e,\phi)}(W)=&(\pm)\frac{A_{J-\frac{1}{2}}{}^{(e,\phi)}}{(2k^2)^{J-\frac{1}{2}}}+(W\mp m)\frac{B_{J-\frac{1}{2}}{}^{(e,\phi)}}{(2k^2)^{J-\frac{1}{2}}}\\&+2(E\mp m)^2\left[\mp\frac{A_{J+\frac{1}{2}}{}^{(e,\phi)}}{(2k^2)^{J+\frac{1}{2}}}+(W\pm m)\frac{B_{J+\frac{1}{2}}{}^{(e,\phi)}}{(2k^2)^{J+\frac{1}{2}}}\right],\end{aligned}\tag{3.2}$$

$$h_{J\pm\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e,\phi)}(W)=-\frac{16\pi W}{k}\frac{a_{J\pm\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e,\phi)}}{(2k^2)^{J-\frac{1}{2}}},\tag{3.3}$$

and

$$\begin{aligned}h_{J\pm\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e,\phi)}(W)=&\frac{WC_{J-\frac{1}{2}}{}^{(e,\phi)}}{(2k^2)^{J-\frac{1}{2}}}\pm\frac{D_{J-\frac{1}{2}}{}^{(e,\phi)}}{(2k^2)^{J-\frac{1}{2}}}\\&+2Wk^2\frac{C_{J+\frac{1}{2}}{}^{(e,\phi)}}{(2k^2)^{J+\frac{1}{2}}}\mp\frac{(2k^2)D_{J+\frac{1}{2}}{}^{(e,\phi)}}{(2k^2)^{J+\frac{1}{2}}}.\end{aligned}\tag{3.4}$$

It is easy to see from expressions (3.1) through (3.4) that these eight partial-wave amplitudes h defined for complex J have no kinematical singularities in the W plane, and their singularity structure in the W plane is precisely the same as that of the functions $h_{l\pm}(W)=[W/(E+m)]f_{l\pm}$ introduced and discussed by Frazer and Fulco.[4]

Besides having nice analytic structure in the W plane, these new amplitudes also have some very simple symmetry properties in the W plane.[5] We have

$$h_{J-\frac{1}{2}\,J-\frac{1}{2}}{}^{J(e,\phi)}(W)=-h_{J+\frac{1}{2},J+\frac{1}{2}}{}^{J(e,\phi)}(-W),\tag{3.5}$$

and

$$h_{J+\frac{1}{2},J-\frac{1}{2}}{}^{J(e,\phi)}(W)=-h_{J-\frac{1}{2},J+\frac{1}{2}}{}^{J(e,\phi)}(-W).\tag{3.6}$$

[4] S. Frautschi and D. Walecka, Phys. Rev. **120**, 1486 (1960); W. Frazer and J. Fulco, *ibid.* **119**, 142.

[5] The author is indebted to Dr. N. Dombey for a discussion about applying the Regge method to nucleons, as an after effect of which the present author was led to realize the importance of the symmetry relation (3.5). However, the present way of applying the Regge method to fermions is different from Dr. Dombey's method. The author is also thankful to Professor M. Gell-Mann

The analog of symmetry relation (3.5) for $a_{J-\frac{1}{2},J-\frac{1}{2}}(W)$ was first pointed out by MacDowell for physical J values.[6] Here it is seen to hold true for complex J values also. The other symmetry relation is new. These relations essentially follow from the reflection properties of the $f_i(W)$'s given by

$$f_1(W)=-f_2(-W), \quad (3.7)$$

and

$$f_3(W)=-f_4(-W). \quad (3.8)$$

The reflection properties (3.7) and (3.8) follow from the invariant nature under $W\to -W$ transformation of the amplitudes A, B, C, and D.

The symmetry properties (3.5) and (3.6) are very significant. We know from them that if $h_{J+\frac{1}{2},J+\frac{1}{2}}{}^{J(e,\phi)}(W)$ has a singularity in the J plane, given by $J=\alpha(W)$, then $h_{J-\frac{1}{2},J-\frac{1}{2}}{}^{J(e,\phi)}(W)$ would have a corresponding singularity in the J plane at $J=\alpha(-W)$. In particular this singularity may be a Regge pole, $J=\alpha(W)$. The symmetry relation also implies the relations between the residue of the Regge pole $J=\alpha(W)$ in $h_{J+\frac{1}{2},J+\frac{1}{2}}{}^{J(e,\phi)}(W)$ and the residue of the Regge pole $J=\alpha(-W)$ in $h_{J-\frac{1}{2},J-\frac{1}{2}}{}^{J(e,\phi)}(W)$.

In the case of parity nonconservation, which we are considering, all the four partial-wave amplitudes having the same J parity and corresponding to the same J are coupled to one another, and thus would share the same Regge poles in the J plane. This sharing property combined with the above symmetry property implies that if one of the amplitudes has a pole at $J=\alpha(W)$, then it would also have a pole at $J=\alpha(-W)$, and the other three amplitudes likewise would have poles at $J=\alpha(W)$ and $J=\alpha(-W)$.

For the real physical case of the conserved parity, we have

$$h_{J\pm\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e,\phi)}(W)=0.$$

Further, unitarity no longer couples the even and odd space-parity parts. Unitarity condition in the physical elastic region reads, for real J and real W,

$$\mathrm{Im}h_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e)}=\frac{k(E\pm m)(2k^2)^{J-\frac{1}{2}}}{16\pi W}|h_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e)}|^2,$$

and

$$\mathrm{Im}h_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^{J(\phi)}=\frac{k(E\pm m)(2k^2)^{J-\frac{1}{2}}}{16\pi W}|h_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^{J(\phi)}|^2;$$

i.e., the four amplitudes $h_{J\mp\frac{1}{2},J\mp\frac{1}{2}}{}^{J(e,\phi)}$ are all decoupled, and, in general, would have different J-plane singularities, apart from the correlation implied by the symmetry relation (3.5) and discussed before.

A family of Regge trajectories can thus be specified if we give the J parity and space parity. So far we did not consider isospin. The inclusion of isospin gives one more quantum number, $I=1/2$, $3/2$.

If we regard the observed particle and resonance states with baryon number one as Regge poles in the πN scattering channel, then they can be interpreted as follows:

(1) Nucleon, isospin one-half, and $F_{\frac{5}{2}}\,\pi N$ resonance with $I=1/2$ at 1680 MeV energy may be regarded as the first two members of the Regge family with $I=1/2$, even parity, and even J parity. It must be observed that without the notion of J parity it would not have been possible to explain the absence of an $I=1/2$, $P_{\frac{3}{2}}\,\pi N$ resonance. Further, both these objects have to lie on the same Regge trajectory; otherwise we would expect to find another particle with nucleon quantum numbers and mass occurring where this Regge trajectory crossed $J=1/2$. We can get an idea of the average slope of this Regge trajectory in terms of the observed masses of N and $F_{\frac{5}{2}}\,\pi N$ resonance. This turns out to be $d\alpha/dW\approx(370\text{ MeV})^{-1}$.

(2) The $D_{\frac{3}{2}}\,\pi N$ resonance with $I=1/2$ at 1510 MeV has to be regarded as the first member of the Regge family with $I=1/2$, odd parity, and odd J parity. An observation of a second member of this family depends on whether this Regge trajectory ever crosses $J=1/2$. On the basis of the above estimate of slope, we might expect it to happen around 2250 MeV, if one were allowed such an extrapolation. More likely, however, is that this is the only observable member of the family.

(3) The 3, 3 resonance (i.e., $P_{\frac{3}{2}}\,\pi N$ resonance with $I=3/2$ at mass 1238 MeV) has to be regarded as the first member of the family with $I=3/2$, even parity, and odd J parity. Using our previous estimate of slope, one would expect this trajectory to exhibit its second member $F_{7/2}\,\pi N$ resonance with $I=3/2$ around 1900 MeV, where one has observed a bump in $I=3/2$ state. The quantum numbers of the bump, however, are not yet certain.

IV. HIGH-ENERGY BACKWARD πN SCATTERING

The results obtained in the last two sections about the J-plane analyticity in the πN scattering (i.e., the s channel) apply equally to the u channel, as this is also a πN scattering channel. As the Regge poles in the u channel control the high-energy backward πN scattering, we are now in a position to give expressions for the πN backward-scattering angular distribution expected in the Regge picture.

We have

$$\frac{d\sigma}{d\Omega}(\pi^\pm p\to\pi^\pm p)=|f_1{}^{(+)}\mp f_1{}^{(-)}-[f_2{}^{(+)}\mp f_2{}^{(-)}]|^2-\frac{1}{k^2}\left[u-\frac{(m^2-1)^2}{s}\right]\times\mathrm{Re}[f_1{}^{(+)}\mp f_1{}^{(-)}]^*[f_2{}^{(+)}\mp f_2{}^{(-)}]. \quad (4.1)$$

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for pointing out that Gribov and Pomeranchuk have also reached similar conclusions. The present discussion makes it clear that the occurrence of two correlated Regge trajectories persists, even when parity is not conserved.

[6] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).

Using crossing symmetry, we have

$$\frac{d\sigma}{d\Omega}(\pi^{\pm}p\to\pi^{\pm}p) = |f_1^{(+)c}\pm f_1^{(-)c}-[f_2^{(+)c}\pm f_2^{(-)c}]|^2 - \frac{1}{k^2}\left[u-\frac{(m^2-1)^2}{s}\right]\mathrm{Re}[f_1^{(+)c}\pm f_1^{(-)c}]^* \times[f_2^{(+)c}\pm f_2^{(-)c}], \quad (4.2)$$

since

$$f_{1,2}^{(+)c}(u,s,t)\pm f_{1,2}^{(-)c}(u,s,t) = f_{1,2}^{(+)}(s,u,t)\mp f_{1,2}^{(-)}(s,u,t). \quad (4.3)$$

This simply expresses the fact that π^+p scattering in the direct channel looks like π^-p scattering in the u channel and vice versa. The superscript c refers to the amplitudes in the crossed u channel with u as energy square.

As the detailed expressions are long, let us illustrate how to work out the contribution of the different Regge poles to $f_1^{(+)c}(u,s,t)$ and to $f_2^{(+)c}(u,s,t)$ by taking nucleon Regge poles as an example.

Now we have

$$f_{1,2}^{(+)c}(u,s,t)=\tfrac{1}{3}f_{1,2}^{(\frac{1}{2})c}(u,s,t)+\tfrac{2}{3}f_{1,2}^{(\frac{3}{2})c}(u,s,t),$$

and

$$f_{1,2}^{(-)c}(u,s,t)=\tfrac{1}{3}f_{1,2}^{(\frac{1}{2})c}(u,s,t)-\tfrac{1}{3}f_{1,2}^{(\frac{3}{2})c}(u,s,t),$$

where superscripts $\frac{1}{2}$, $\frac{3}{2}$ refer to the value of total isospin. As the nucleon Regge trajectory has $I=\frac{1}{2}$, it would not contribute to $I=\frac{3}{2}$ amplitudes and we get

$$[f_{1,2}^{(+)c}+f_{1,2}^{(-)c}]_N=\tfrac{2}{3}[f_{1,2}^{(\frac{1}{2})c}(u,s,t)]_N,$$

and

$$[f_{1,2}^{(+)c}-f_{1,2}^{(-)c}]_N=0,$$

where the subscript N stands for nucleon Regge contribution.

The physical nucleon pole appears in the amplitude $h_{J+\frac{1}{2},J+\frac{1}{2}}{}^{J(e)}(W_u)$ at $J=\frac{1}{2}$ and $W_u=m$ in the u channel, where $u=W_u^2$ and W_u is the c.m. energy in the u channel. If we denote this Regge trajectory by $J=\alpha_N(W_u)$, the amplitude $h_{J-\frac{1}{2},J-\frac{1}{2}}{}^{J(e)}(W_u)$ will have a trajectory given by $J=\alpha_N(-W_u)$. Using the Sommerfeld-Watson transform for f_1, f_2 in the u channel, obtained by replacing W and z by W_u and z_c (the energy and cosine of the angle of scattering in the u channel) in expression (2.14) and using the symmetry relation (3.5), we find that the contribution of the nucleon Regge pole to $f_1^{(c)}$, $f_2^{(c)}$ is given by

$$[f_{1,2}^{(\frac{1}{2})c}(u,s,t)]_N = \frac{(E_u-m)(2k_u^2)^{\alpha_N(W_u)-\frac{1}{2}}b_N(W_u)}{32W_u\cos[\pi\alpha_N(W_u)]} \times[P_{\alpha_N(W_u)\mp\frac{1}{2}}'(z_c)\mp P_{\alpha_N(W_u)\mp\frac{1}{2}}'(-z_c)] - \frac{(E_u+m)(2k_u^2)^{\alpha_N(-W_u)-\frac{1}{2}}b_N(-W_u)}{32W_u\cos[\pi\alpha_N(-W_u)]} \times[P_{\alpha_N(-W_u)\pm\frac{1}{2}}'(z_c)\pm P_{\alpha_N(-W_u)\pm\frac{1}{2}}(-z_c)],$$

where

$$b_N(W_u)=\lim_{J\to\alpha_N(W_u)}\{[\alpha_N(W_u)-J]h_{J+\frac{1}{2},J+\frac{1}{2}}{}^{J(e)}(W_u)\},$$

$$4k_u^2=u-2m^2-2+(m^2-1)^2/u,$$

$$E_u=(W_u^2+m^2-1)/2W_u,$$

and

$$z_c=-[s-m^2-1+2E_u(W_u-E_u)]/2k_u^2. \quad (4.4)$$

By substituting Eq. (4.4) together with similar contributions from other Regge poles in Eq. (4.2) we have the angular distribution in the backward πp scattering.

Now the backward direction in the s-channel πN scattering is given by

$$u-(m^2-1)^2/s=0.$$

Thus, at very high energies, the backward cone has the u values, which are negative; i.e., W_u is pure imaginary. If $\alpha_N(W_u)$ and $b_N(W_u)$ are real analytic functions with cuts on the real axis only, then $\alpha_N(-W_u)$ and $b_N(-W_u)$ would be complex conjugates of $\alpha_N(W_u)$ and $b_N(W_u)$, respectively, and there would be interference terms between the trajectories $J=\alpha_N(W_u)$ and $J=\alpha_N(-W_u)$, which would lead to oscillations in the angular distribution.

V. REGGE POLES IN THE $\pi+\pi\to N+\bar{N}$ CHANNEL

We now come to a discussion of the J-plane analyticity in the $\pi+\pi\to N+\bar{N}$ channel; i.e., the t channel. The Regge poles in this channel control the high-energy forward elastic πN scattering.

The partial-wave decomposition in this channel is given by[7]

$$A^{(\pm)}(s,u,t)=\frac{8\pi}{p^2}\sum(J+\tfrac{1}{2})(pq)^J \times\left[\frac{m\cos\theta_3}{[J(J+1)]^{1/2}}P_J'(\cos\theta_3)f_-^{(\pm)J}(t) - P_J(\cos\theta_3)f_+^{(\pm)J}(t)\right], \quad (5.1)$$

and

$$B^{(\pm)}(s,u,t)=8\pi\sum\frac{(J+\frac{1}{2})}{[J(J+1)]^{1/2}} \times(pq)^{J-1}P_J'(\cos\theta_3)f_-^{(\pm)J}(t), \quad (5.2)$$

where

$$t=4(p^2+m^2)=4(q^2+1),$$

$$\cos\theta_3=(s+p^2+q^2)/(2pq)=z_t,$$

$f_{\pm}^{(\pm)J}$ = same definition as of Frazer and Fulco,

and the sums over J run through $J=0, 2, 4, \cdots$ for $A^{(+)}$ and $B^{(+)}$, that is $I=0$; and $J=1, 3, \cdots$ for $A^{(-)}$ and $B^{(-)}$, that is $I=1$.

[7] W. Frazer and J. Fulco, Phys. Rev. **117**, 1603 (1960).

In what follows we do not consider the analytic continuation of $f_{\pm}^{(\pm)J}$ into the complex J plane, but rather the continuation of $f_{+}^{(\pm)J}$ and $(\mathfrak{f})_{-}^{(\pm)J}$ defined by

$$(\mathfrak{f})_{-}^{(\pm)J}=\frac{J+\frac{1}{2}}{[J(J+1)]^{1/2}}f_{-}^{(\pm)J}, \tag{5.3}$$

as these are the quantities that we always encounter. This gets rid of the fixed branch points in J at $J=0, -1$. We have for physical J values

$$f_{+}^{(\pm)J}=\frac{1}{8\pi}\left\{-p^2\frac{A_J^{(\pm)}}{(pq)^J}+\frac{m}{(2J+1)(pq)^{J-1}}\times[(J+1)B_{J+1}^{(\pm)}+JB_{J-1}^{(\pm)}]\right\}. \tag{5.4}$$

$$(\mathfrak{f})_{-}^{(\pm)J}=\frac{B_{J-1}^{(\pm)}-B_{J+1}^{(\pm)}}{16\pi(pq)^{J-1}}.$$

Using these expressions to project out these partial waves, we obtain, after certain simplifications,

$$f_{+}^{(\pm)J}(t)=-\frac{p^2}{8\pi^2}\frac{[1\pm(-)^J]}{(pq)^{J+1}}\int ds'\times\left[A_s^{(\pm)}(s',t)-\frac{s'+p^2+q^2}{2p^2}B_s^{(\pm)}(s',t)\right]\times Q_J\left(\frac{s'+p^2+q^2}{2pq}\right),$$

and (5.5)

$$(\mathfrak{f})_{-}^{(\pm)J}=\frac{1}{16\pi^2}\frac{[1\pm(-)^J]}{(pq)^J}\int ds'\,B_s^{(\pm)}(s',t)\times\left[Q_{J-1}\left(\frac{s'+p^2+q^2}{2pq}\right)-Q_{J+1}\left(\frac{s'+p^2+q^2}{2pq}\right)\right].$$

We have used the crossing symmetry (Bose statistics for the pions) also in writing these expressions. Looking at these expressions, one again sees that, apart from the factor $[1\pm(-)^J]$, the quantities $f_{+}^{(\pm)J}$ and $(\mathfrak{f})_{-}^{(\pm)J}$ define analytic continuations that are suitable for making Sommerfeld-Watson transforms. Thus we again define the even and odd J-parity continuations by replacing $(-)^J$ by $+1$ for even J parity and by -1 for odd J-parity continuations. This makes the odd J-parity continuations for $I=0$ and the even J-parity continuations for $I=1$ identically vanish. This is a particular instance in which a symmetry property (here, Bose statistics for pions) tells us that only one J parity is physical. Since only one of the J-parity continuations is nonzero, we shall use the same notation as $f_{+}^{(\pm)J}$ and $(\mathfrak{f})_{-}^{(\pm)J}$ to denote the nonvanishing one.

The analytic properties in t of the J-plane analytic continuations $f_{+}^{(\pm)J}$ and $(\mathfrak{f})_{-}^{(\pm)J}$ are precisely the same as that of the physical partial waves. They are thus real analytic functions in the t plane with a right-hand cut $4<t<\infty$, and a left-hand cut $-\infty<t<4(1-1/4m^2)$, on the real axis.

Since unitarity couples both f_{+}^{+J} and $(\mathfrak{f})_{-}^{+J}$ to a number of common channels like the $I=0$, $\pi\pi$ scattering channel, they will share the same Regge poles together with the $I=0$, $\pi\pi$ scattering amplitude. Similarly for $(f_{+})^{-J}$ and $(\mathfrak{f})_{-}^{-J}$.

VI. HIGH-ENERGY FORWARD ELASTIC SCATTERING

High-energy forward elastic scattering is dominated by the Regge poles in the crossed channel $\pi+\pi\rightarrow N+\bar{N}$; i.e., the t channel, which we analyzed in the last section from the Regge point of view.

We have for differential and total cross sections,

$$d\sigma/d\Omega=|f_1+f_2|^2+(t/k^2)\,\mathrm{Re}f_1^*f_2, \tag{6.1}$$

and

$$\sigma^{\mathrm{total}}=\frac{4\pi W}{m(\omega^2-1)^{\frac{1}{2}}}\,\mathrm{Im}(f_1+f_2)_{t=0}, \tag{6.2}$$

with

$$\omega=(s-m^2-1)/(2m)=\text{the lab energy of the pion.}$$

Here one has to substitute proper isospin combinations for f_1 and f_2. Thus

$$f_i=f_i^{(+)}\pm f_i^{(-)}\quad\text{for}\quad \pi^{\mp}p\rightarrow\pi^{\mp}p,$$

and

$$f_i=-\sqrt{2}f_i^{(-)}\quad\text{for}\quad \pi^{-}p\rightarrow\pi^0 n. \tag{6.3}$$

Re-expressing Eqs. (6.1) and (6.2) in terms of amplitudes A' and B, where

$$A'=A+\frac{\omega+t/(4m)}{1-t/(4m^2)}B, \tag{6.4}$$

we obtain,

$$\frac{d\sigma}{d\Omega}=\left(\frac{m}{4\pi W}\right)^2\left[\left(1-\frac{t}{4m^2}\right)|A'|^2+\frac{t}{4m^2}\left(s-\frac{(m+\omega)^2}{1-t/(4m^2)}\right)|B|^2\right], \tag{6.5}$$

and

$$\sigma^{\mathrm{total}}=\frac{1}{(\omega^2-1)^{\frac{1}{2}}}\,\mathrm{Im}A'(s;\,t=0). \tag{6.6}$$

Now we have from Eqs. (5.1), (5.2), and (5.3),

$$A'^{(\pm)}(s,t)=-\frac{8\pi}{p^2}\sum_J(pq)^J(J+\tfrac{1}{2})f_{+}^{(\pm)J}(t)P_J(\cos\theta_3),$$

and

$$B^{(\pm)}(s,t)=8\pi\sum_J(pq)^{J-1}(\mathfrak{f})_{-}^{(\pm)J}P_J'(\cos\theta_3). \tag{6.7}$$

On the hypothesis that large s (i.e., $\cos\theta_3$) behavior is dominated by the Regge poles in the t channel, we have

$$A'^{(\pm)} \underset{s\to\infty}{\longrightarrow} \frac{2\pi i}{p^2}\left(\frac{pq}{m}\right)^{\alpha^\pm(t)} [\alpha^\pm(t)+\tfrac{1}{2}]b_+{}^\pm(t) \times\left[\frac{P_{\alpha^\pm(t)}(-z_t)\pm P_{\alpha^\pm(t)}(z_t)}{\sin\pi\alpha^\pm(t)}\right], \quad (6.8)$$

and

$$B^{(\pm)} \underset{s\to\infty}{\longrightarrow} +2\pi i\left(\frac{pq}{m}\right)^{\alpha^\pm(t)-1} b_-{}^\pm(t) \times\left[\frac{P_{\alpha^\pm(t)}{}'(-z_t)\mp P_{\alpha^\pm(t)}{}'(z_t)}{\sin\pi\alpha^\pm(t)}\right],$$

where $\alpha^+(t)$ and $\alpha^-(t)$ are Regge poles that have maximum real parts for the isospin zero, $\pi+\pi\to N+\bar{N}$ channel, and for the isospin one, $\pi+\pi\to N+\bar{N}$ channel, respectively, and where

$$b_+{}^\pm(t)=\lim_{J\to\alpha^\pm(t)}\{m^J f_+{}^{\pm(J)}(t)[J-\alpha^\pm(t)]\},$$

and

$$b_-{}^\pm(t)=\lim_{J\to\alpha^\pm(t)}\{m^{J-1}(\mathfrak{f})_-{}^{\pm(J)}(t)[J-\alpha^\pm(t)]\}. \quad (6.8')$$

In writing these expressions we have used the results concerning the J parity and sharing of Regge poles by different amplitudes, which were established in the last section.

These expressions (6.8) could be further simplified to

$$A'^{(\pm)} \underset{s\to\infty}{\longrightarrow} C_+{}^\pm(t)\left(\frac{s}{2m}\right)^{\alpha\pm(t)}\left[\frac{1\pm e^{-i\pi\alpha\pm(t)}}{\sin\pi\alpha^\pm(t)}\right],$$

and (6.9)

$$B^{(\pm)} \underset{s\to\infty}{\longrightarrow} \alpha^\pm(t)C_-{}^\pm(t)\left(\frac{s}{2m}\right)^{\alpha\pm(t)-1}\left[\frac{1\pm e^{-i\pi\alpha\pm(t)}}{\sin\pi\alpha^\pm(t)}\right],$$

where $C_+{}^\pm(t)$ is linearly related to $b_+{}^\pm(t)$, and $C_-{}^\pm(t)$ to $b_-{}^\pm(t)$. Substituting these behaviors (6.9) into our expressions for total cross sections, we get

$$\sigma^{\text{total}}(\pi^-p)+\sigma^{\text{total}}(\pi^+p) \underset{s\to\infty}{\longrightarrow} C_+{}^+(0)\left(\frac{s}{2m}\right)^{\alpha^+(0)-1},$$

and (6.10)

$$\sigma^{\text{total}}(\pi^-p)-\sigma^{\text{total}}(\pi^+p) \underset{s\to\infty}{\longrightarrow} C_+{}^-(0)\left(\frac{s}{2m}\right)^{\alpha^-(0)-1}.$$

Now if the constancy and equality of the π^+p and π^-p cross section is to be achieved in this picture, then we must have

$$\alpha^+(0)=1,$$

and (6.11)

$$\alpha^-(0)<1.$$

Thus these must be a trajectory having zero baryon number, even G parity, even J parity, and zero isospin; i.e., the trajectory has the quantum numbers of the vacuum that must pass through 1 at $t=0$. This is the Pomeranchuk trajectory. There cannot be any trajectory that passes through a point $J>1$ at $t=0$ otherwise we would have a cross section increasing as a power of energy which is certainly not allowed by the Mandelstam representation. Also, for isospin one we expect the ρ Regge trajectory to be same as $\alpha^-(t)$. Now $\text{Re}\,\alpha^-(t)=1$ at $t\approx 30m_\pi{}^2$; hence, at $t=0$ we would automatically have $\alpha^-(0)<1$.

Thus, at high energies, the π^+p and π^-p scattering will both be dominated by the Pomeranchuk-Regge pole and we will have

$$\frac{d\sigma}{dt}(\pi^\pm p\to\pi^\pm p) \underset{s\to\infty}{\longrightarrow} \frac{1}{16\pi}\left(\frac{s}{2m}\right)^{2[\alpha^+(t)-1]} \times\left\{|c_+{}^+(t)|^2-\frac{t}{4m^2}(|c_+{}^+(t)|^2+|\alpha^+(t)c_-{}^+(t)|^2)\right\} \times\left|\frac{1+e^{-i\pi\alpha^+(t)}}{\sin\pi\alpha^+(t)}\right|^2. \quad (6.12)$$

However, for the charge exchange $\pi^-p\to\pi^0n$, the Pomeranchuk-Regge pole cannot contribute because there would have to be a charge exchange in the crossed channel; this cannot happen because the Pomeranchuk trajectory has zero isospin. Charge exchange is a pure $I=1$ process when looked at in the t channel. Thus, this process is dominated by the ρ Regge pole, and we have

$$\frac{d\sigma}{dt}(\pi^-p\to\pi^0n) \underset{s\to\infty}{\longrightarrow} \frac{1}{8\pi}\left(\frac{s}{2m}\right)^{2[\alpha^-(t)-1]} \times\left\{|c_+{}^-(t)|^2-\frac{t}{4m^2}(|c_+{}^-(t)|^2+|\alpha^-(t)c_-{}^-(t)|^2)\right\} \times\left|\frac{1-e^{-i\pi\alpha^-(t)}}{\sin\pi\alpha^-(t)}\right|^2. \quad (6.13)$$

By using these expressions (6.12) and (6.13), it should be experimentally possible to determine the Pomeranchuk and ρ trajectories for negative values of t. A significant feature of the Regge-pole hypothesis is the logarithmic shrinkage of the width of the diffraction peak with energy.

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APPENDIX. THE RANGE OF THE EXCHANGE POTENTIAL

There has been some uncertainty as to what quantity should properly be called the range of the exchange potential in the case of the scattering of two unequal mass particles, such as πN scattering. The discussion in Sec. II clarifies this situation.

It will be seen from expressions (2.12) and (2.13) that the absorptive parts in the t and u channels having the same value of the integration variable x' superimpose each other. Now $x'=t$ for t absorptive parts and $x'=u-(m^2-1)^2/s$ for u absorptive parts. Hence the range of the exchange force arising from the exchange of mass $\sqrt{u}$ is $[u-(m^2-1)^2/s]^{-1/2}$ in the sense that $(t)^{-\frac{1}{2}}$ is the range of the direct force arising from an exchange of mass $\sqrt{t}$ in the t channel. Unlike the direct force, the range of the exchange force is energy dependent and gets smaller as the energy gets larger. In particular, the exchange of a single nucleon gives rise at low energies to a force of range of approximately $(2m)^{-1/2}$ and approaches the naively expected range $(m)^{-1}$ only at very high energy.

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Fluctuation Compressibility Theorem and Its Application to the Pairing Model

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A theorem of statistical mechanics relates density fluctuations to compressibility. A new derivation of this is given. The theorem is violated in the BCS model of a superconductor. The difficulty is resolved by those same improvements in the theory which lead to a gauge-invariant Meissner effect.

I. INTRODUCTION

It has been observed by Lüders[1] that density fluctuations in the BCS model of a superconductor violate a standard result of statistical mechanics (Sec. 2, Sec. 3). The difficulty is analyzed here. It is found to be resolved, at least for zero temperature, by those same improvements of the theory which lead to a gauge-invariant Meissner effect (Sec. 4, Sec. 5). A new derivation of the standard theorem is given (Sec. 6).

II. THEOREM

Consider an infinite homogeneous system in thermal equilibrium, specified by temperature T and chemical potential μ. The two-particle correlation function is defined by

$$G(\mathbf{x}-\mathbf{y})=\langle\rho(\mathbf{x})\rho(\mathbf{y})\rangle-\langle\rho(\mathbf{x})\rangle\langle\rho(\mathbf{y})\rangle, \tag{1}$$

where $\rho(\mathbf{x})$ is density at position $\mathbf{x}$, and brackets $\langle\ \rangle$ denote thermal averaging. The standard result[2] is that

$$\int d\mathbf{x}\, G(\mathbf{x})=kT\frac{\partial\rho}{\partial\mu}, \tag{2}$$

where ρ is mean density. An equivalent statement is that in a large subvolume Ω' the fluctuation of particle number

$$N'=\int_{\Omega'} d\mathbf{x}\,\rho(\mathbf{x})$$

is given by

$$\langle N'^2\rangle-\langle N'\rangle^2=\Omega' kT(\partial\rho/\partial\mu), \tag{3}$$

or with a different form of the right-hand side

$$\langle N'^2\rangle-\langle N'\rangle^2=\Omega'\rho kT(\partial\rho/\partial p), \tag{4}$$

where p is pressure.

The usual argument is that for large enough Ω' one can ignore interaction across the dividing surface with the remainder of the system. The latter is treated merely as a reservoir of particles. The subsystem in Ω' is then represented, to some unspecified degree of accuracy, by a grand canonical ensemble. Equation (3) is readily derived, and (2) follows from it.

The theorem has been stated for an infinite system. In formal discussion one considers first a large but finite system, of volume Ω. We then use the conventional periodic boundary conditions, so that the quantity on the right-hand side of (1) remains a function only of $(\mathbf{x}-\mathbf{y})$. It is essential that the limit $\Omega\rightarrow\infty$ is taken *before* the integration in (2) is performed. It is easily seen that the quantity

$$\lim_{\Omega\to\infty}\int_{\Omega} d\mathbf{x}\, G(\mathbf{x})$$

is ensemble dependent. In fact, it is proportional to the

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[1] G. Lüders (unpublished).
[2] See for example L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, London, 1958), p. 365.